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Plane contact problem for a layer involving frictional heating

V.J. Pauk

Department of Geotechnics and Structural Engineering, Łódź University of Technology, Łódź, al. Politechniki 6, PL-90-538, Poland

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Abstract

Plane steady contact problems of a rigid insulated cylinder sliding over a thermoelastic layer is solved. The heat generation in the contact region is caused by friction forces. The problem is investigated by the method of integral equations. An exponential series approximation is used for the evaluation of the kernels of integral equations. Numerical results are presented in diagrams.

1. Introduction

The study of the heat generation in the rubbing contact of solids can be found in many papers. The plane steady problems involving frictional heating for a thermoelastic half-space were investigated in refs. [1–3]. But in many cases the modelling of components of real frictional couples by a half-space is impossible. Many pairs can be often considered as a layer. The elastic contact problems for a layer was investigated in ref. [4]. Plane thermoelastic contact problems for a layer involving frictional heating is studied in this paper. The problem is explored by the method of integral equations. The numerical analysis is done to study the effect of the layer thickness on the level of generated temperature and on the limit value of contact zone. The result obtained is compared with the solution of the analogous problem for a half-space.

2. Problem formulation

The geometry of the contact problem is shown in Fig. 1. The rigid thermoinsulated cylinder (punch) of the radius R is pressed by the load P to the upper surface of thermoelastic layer of the thickness H and slides with the constant velocity V in z-direction. The lower surface of the layer is bounded with a rigid base. The friction force $\sigma_{zy}(x)$ is generated in the contact region $(-a, a)$. It is connected with the contact pressure $p(x)$ by the Amontons law

$$\sigma_{zy}(x) = fp(x) \tag{1}$$

where $f =$ const is the friction coefficient. It is supposed that the heating in the contact region is produced by the friction force. The generated heat is conducted into the thermoelastic layer only. It is assumed that the thermal process in the layer is steady. Convective radiation occurs at the free boundary of the layer. The problem is considered to be planar.

Mathematically, the problem formulated above is reduced to the solution of the thermoelasticity equations

$$2(1-v)\frac{\partial^2 u}{\partial x^2} + (1-2v)\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial x\,\partial y} = 2(1+v)\alpha\frac{\partial T}{\partial x} \tag{2}$$

$$(1-2)\frac{\partial^2 v}{\partial x^2} + 2(1-v)\frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 u}{\partial x\,\partial y} = 2(1+v)\alpha\frac{\partial T}{\partial y} \tag{3}$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{4}$$

with the following boundary conditions

$$K\frac{\partial T}{\partial y} = q(x), \quad |x| \leqslant a, \quad y = 0 \tag{5}$$

$$K\frac{\partial T}{\partial y} = -hT(x,y), \quad |x| > a, \quad y = 0 \tag{6}$$

$$K\frac{\partial T}{\partial y} = h_0 T(x,y), \quad |x| < \infty, \quad y = -H \tag{7}$$

$$\sigma_{yy}(x,y) = -p(x), \quad |x| \leqslant a, \quad y = 0 \tag{8}$$

$$\sigma_{yy}(x,y) = 0, \quad |x| > a, \quad y = 0 \tag{9}$$

$$\sigma_{xy}(x,y) = 0, \quad |x| < \infty, \quad y = 0 \tag{10}$$

$$u(x,y) = v(x,y) = 0, \quad |x| < \infty, \quad y = -H \tag{11}$$

$$q(x) = -V\sigma_{xy}, \quad |x| \leqslant a, \quad y = 0 \tag{12}$$

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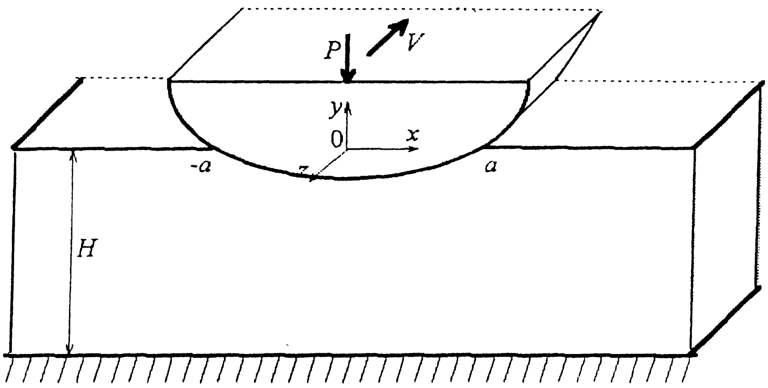

Fig. 1. Geometry of contact problem.

$$\frac{\partial v(x,y)}{\partial x} = -\frac{x}{R}, \quad |x| \leqslant a, \quad y=0 \tag{13}$$

where u, v are displacements; σ_{xx}, σ_{xy}, σ_{yy} are stresses; T is temperature field; p is contact pressure; q is heat flux; ν, α, K are, respectively, Poisson's ratio, thermal expansion, thermal conductivity; h, h_0 are radiation coefficients at the upper and lower surface of the layer, respectively.

3. System of integral equations

The problems (2)–(13) was solved by using the Fourier transform method. It was obtained that the temperature field in the layer has the following form

$$T(x,y) = \frac{1}{\pi K}\int_{-a}^{a} [hT(x',0)+q(x')]M(x-x',y)\,\mathrm{d}x' \tag{14}$$

where

$$M(z,y) = \int_0^\infty \frac{\cos(\xi z)}{\xi+\gamma} L(\xi,y)\,\mathrm{d}\xi$$

$$L(\xi,y) = \frac{(\xi+\gamma)[(\xi\tanh(\xi H)+\gamma_0)\sinh(\xi y) + (\xi+\gamma_0\tanh(\xi H))\cosh(\xi y)]}{\xi(\gamma+\gamma_0)+(\xi^2+\gamma\gamma_0)\tanh(\xi H)}$$

and

$$\gamma = h/K, \quad \gamma_0 = h_0/K.$$

The vertical displacements of the upper surface of the layer can be presented as the sum of the elastic and thermal displacements

$$\frac{\mathrm{d}v(x)}{\mathrm{d}x} = \frac{\mathrm{d}v^{\mathrm{e}}(x)}{\mathrm{d}x} + \frac{\mathrm{d}v^{\mathrm{th}}(x)}{\mathrm{d}x}. \tag{15}$$

The elastic displacements are determinated from the formula [4]

$$\frac{\mathrm{d}v^{\mathrm{e}}(x)}{\mathrm{d}x} = \frac{1-\nu}{\pi\mu}\int_{-a}^{a} p(x')S(x-x')\,\mathrm{d}x' \tag{16}$$

where

$$S(z) = -\frac{1}{z}+S_1(z)$$

$$S_1(z) = \int_0^\infty [1-W(\xi)]\sin(\xi z)\,\mathrm{d}\xi$$

$$W(\xi) = \frac{2\kappa\sinh(2\xi H)-4\xi H}{2\kappa\cosh(2\xi H)+1+\kappa^2+4\xi^2H^2}, \quad \kappa = 3-4\nu$$

and μ is a shear modulus.

The thermal displacements of the surface are obtained in the forms

$$\frac{\mathrm{d}v^{\mathrm{th}}(x)}{\mathrm{d}x} = \frac{\delta}{\pi}\int_{-a}^{a} [hT(x',0)+q(x')]N(x-x')\,\mathrm{d}x' \tag{17}$$

where

$$N(z) = \int_0^\infty \frac{\sin(\xi z)}{\xi+\gamma} R(\xi)\,\mathrm{d}\xi$$

$$R(\xi) = \frac{(\xi+\gamma)[(\xi\tanh(\xi H)+\gamma_0)d_1(\xi) + (\xi+\gamma_0\tanh(\xi H))d_2(\xi)]}{[\xi(\gamma+\gamma_0)+(\xi^2+\gamma\gamma_0)\tanh(\xi H)] \times [(1-2\nu)^2+\xi^2H^2+\kappa\cosh(\xi H)]}$$

$$d_1(\xi) = \xi^2H^2+(1-2\nu)\sinh^2(\xi H) - 2(1-\nu)\xi H\sinh(2\xi H)$$

$$d_2(\xi) = 2(1-\nu)[0.5\sinh(2\xi H)+\xi H\cosh(2\xi H)]$$

and $\delta = (1+\nu)\alpha/K$ is the thermal distortivity.

The first integral equation of the problem under consideration is obtained from satisfying the boundary condition (13) with the help of relations (15)–(17) and (1), (12). By rewriting of formula (14) at the boundary $y=0$ with the help of relations (1) and (12), the second integral equation is obtained. Thus, the system of two integral equations of the problem has the following form

$$\frac{1-\nu}{\pi\mu}\int_{-a}^{a} p(x')\left[\frac{1}{x'-x}+S_1(x-x')\right]\mathrm{d}x'$$

$$-\frac{fV\delta}{\pi}\int_{-a}^{a} p(x')N(x-x')\,\mathrm{d}x'$$

$$+\frac{\delta h}{\pi}\int_{-a}^{a} T(x')N(x-x')\,\mathrm{d}x' = -\frac{x}{R}, \quad |x| \leqslant a \tag{18}$$

$$T(x)-\frac{h}{\pi K}\int_{-a}^{a} T(x')M(x-x',0)\,\mathrm{d}x'$$

$$+\frac{fV}{\pi K}\int_{-a}^{a} p(x')M(x-x',0)\,\mathrm{d}x' = 0, \quad |x| \leqslant a. \tag{19}$$

The contact pressure must satisfy the equilibrium condition

$$\int_{-a}^{a} p(x')\,\mathrm{d}x' = P \tag{20}$$

and physical conditions

$$p(-a) = p(a) = 0. \tag{21}$$

By introducing new variables and functions

$s = x/a, \quad r = x'/a, \quad p^*(r) = ap(x')/P,$
$T^*(r) = KT(x')/fVP$

the system (18), (19) can be transformed to the dimensionless form (the asterisks are omitted)

$$\frac{1}{\pi}\int_{-1}^{1} p(r)\left[\frac{1}{r-s}+S_1(s-r)-\theta N(s-r)\right]\mathrm{d}r$$

$$+\frac{\theta\, Bi}{\pi}\int_{-1}^{1} T(r)N(s-r)\,\mathrm{d}r$$

$$= -\frac{2}{\pi}\frac{a^2}{a_{\mathrm{H}}^2}\frac{P_{\mathrm{H}}}{P}s, \quad |s| \leqslant 1 \tag{22}$$

$$T(s)-\frac{Bi}{\pi}\int_{-1}^{1} T(r)M(s-r)\,\mathrm{d}r$$

$$+\frac{1}{\pi}\int_{-1}^{1} p(r)M(s-r)\,\mathrm{d}r = 0, \quad |s| \leqslant 1 \tag{23}$$

$$\int_{-1}^{1} p(r)\,\mathrm{d}r = 1 \tag{24}$$

$$p(-1) = p(1) = 0 \tag{25}$$

where

$$\theta = \frac{\alpha f V\mu\delta}{1-\nu}, \quad Bi = \frac{ah}{K} \tag{26}$$

and a_{H}, P_{H} are, respectively, the half-width of the contact pach and the load in Hertz formula [5]

$$a_{\mathrm{H}}^2 = \frac{2P_{\mathrm{H}}R}{\pi}\frac{1-\nu}{\mu}.$$

4. Analysis of the kernels of integral equations

The dimensionless kernels of the system of integral equations (22), (23) can be presented as

$$M(z) = M_0(z)+M_1(z),$$
$$N(z) = N_0(z)+N_1(z)$$
$$M_0(z) = \int_0^\infty \frac{\cos(\zeta z)}{\zeta+Bi}\,\mathrm{d}\zeta,$$
$$N_0(z) = \int_0^\infty \frac{\sin(\zeta z)}{\zeta+Bi}\,\mathrm{d}\zeta$$
$$S_1(z) = \int_0^\infty F_0(\zeta)\sin(\zeta z)\,\mathrm{d}\zeta,$$
$$M_1(z) = \int_0^\infty F_1(\zeta)\cos(\zeta z)\,\mathrm{d}\zeta,$$
$$N_1(z) = \int_0^\infty F_2(\zeta)\sin(\zeta z)\,\mathrm{d}\zeta$$
$$F_0(\zeta) = 1-W(\zeta)$$
$$F_1(\zeta) = \frac{L(\zeta)-1}{\zeta-Bi},$$
$$F_2(\zeta) = \frac{R(\zeta)-1}{\zeta+Bi}. \tag{27}$$

The function $N_0(z)$ is regular and $M_0(z)$ has a logarithmic singularity. They were analysed in the paper [3].

The kernels $S_1(z)$, $M_1(z)$ and $N_1(z)$ depend on the boundary conditions at the free surface of the layer. These kernels are functions of the dimensionless thickness of the layer $\lambda = H/a$ and Biot's coefficient $Bi_0 = ah_0/K$ at the lower surface. The system of integral equations of the corresponding problem for a half-space [3] is obtained from (22), (23) by omitting of the kernels $S_1(z)$, $M_1(z)$ and $N_1(z)$.

The functions $F_k(\zeta)$, $k = 0$, 1, 2 tend to constants at $\zeta \to 0$ and decay exponentially for large ζ. These properties allow us to present these functions in the form of finite sums

$$F_k(\zeta) = \sum_{m=1}^{M} f_m^{(k)}\,\mathrm{e}^{-m\beta_k\zeta}, \quad k = 0,1,2 \tag{28}$$

where M is a finite number and the constants $\beta_k, f_m^{(k)}$ are unknown. For their calculations the squared error method was used. This approach, which was applied in paper [6], is outlined in the Appendix.

By help of formulae (27), (28) the kernels $S_1(z)$, $M_1(z)$ and $N_1(z)$ may be presented in the forms

$$S_1(z) = \sum_{m=1}^{M} f_m^{(0)} \frac{z}{(m\beta_0)^2 + z^2}$$

$$N_1(z) = \sum_{m=1}^{M} f_m^{(1)} \frac{z}{(m\beta_1)^2 + z^2}$$

$$M_1(z) = \sum_{m=1}^{M} f_m^{(2)} \frac{m\beta_2}{(m\beta_2)^2 + z^2}. \quad (29)$$

5. Discretization

The first equation of the system (22), (23) is a Cauchy-type singular integral equation of first kind for the unknown contact pressure expressed in the terms of the temperature and known functions. We present the function $p(r)$ in the form

$$p(r) = \varphi(r)\sqrt{1-r^2}, \quad (30)$$

where $\varphi(r)$ is a regular function. Thus, the conditions (25) are satisfied automatically.

The equation (23) is a Fredholm-type integral equation of second kind for the function $T(r)$. Therefore the unknown temperature is the function of the class of limited functions.

By help of Gauss–Chebyshev quadrature method [7] and rectangle quadratures, the discretized form of system (22)–(24) is obtained

$$\gamma_{0n} + \frac{1}{\pi}\sum_{k=1}^{n} \varphi(r_k) w_k \left[\frac{1}{s_m - r_k} + S_1(s_m - r_k) - \theta N(s_m - r_k)\right]$$

$$+ \frac{\theta\, Bi}{\pi} \sum_{k=1}^{n} \frac{2}{n} T(\rho_k) N(s_m - \rho_k)$$

$$= -\frac{2}{\pi}\frac{a^2}{a_{\mathrm{H}}^2}\frac{P_{\mathrm{H}}}{P} s_m, \quad m = 1, \ldots, n+1 \quad (31)$$

$$T(\rho_m) - \frac{Bi}{\pi}\sum_{k=1}^{n} T(\rho_k) A_{km} + \frac{1}{\pi}\sum_{k=1}^{n} w_k \varphi(r_k) M(\rho_m - r_k) = 0,$$

$$m = 1, \ldots, n \quad (32)$$

$$\sum_{k=1}^{n} w_k \varphi(r_k) = 1 \quad (33)$$

where

$$r_k = \cos\left[\frac{\pi k}{n+1}\right], \quad w_k = \frac{1}{n+1}, \quad k = 1, \ldots, n$$

$$s_m = \cos\left[\frac{\pi}{2}\frac{2m-1}{n+1}\right], \quad m = 1, \ldots, n+1$$

$$\rho_k = -1 + 2(k-0.5)/n, \quad k = 1, \ldots, n$$

$$A_{km} = \int_{X_2}^{X_1} M(u)\,\mathrm{d}u = \frac{\pi}{2\,Bi}[\mathrm{sign}(X_1)$$

$$-\mathrm{sign}(X_2)] - \frac{1}{Bi}[N_0(X_1) - N_0(X_2)]$$

$$+ \sum_{\mu=1}^{M} f_\mu^{(2)} [A\tan(X_1/\mu\beta_2) - A\tan(X_2/\mu\beta_2)]$$

$$X_1 = 2(m-k+0.5)/n,$$

$$X_2 = 2(m-k-0.5)/n, \quad k, m = 1, \ldots, n.$$

The introduction of the regularized constant γ_{0n} in equations (31) provides the existence of the solution [7].

6. Results and discussion

The dimensionless thickness λ, Biot's coefficients Bi, Bi_0 and the parameter θ are given. Note, that the semi-width of the contact region a, i.e. the ratio a/a_{H} is unknown. We assume that the contact region is equal to that in the Hertz problem for half-space (i.e. $a/a_{\mathrm{H}} = 1$), but the ratio P_{H}/P, which is needed to obtain this fixed region must be found.

The numerical analysis was done to investigate the effect of the layer thickness on the generated temperature, on the limit value of the contact region size, as well as to compare the obtained results with the solution of the corresponding problem for a half-space.

Figure 2 shows the effect of the thickness γ on the dimensionless temperature at the surface $y = 0$ for $\theta = 0.5$. The rise of the convective radiation leads to the decrease of the temperature level. The growth of the layer thickness causes the lowering of the temperature and the results for the half-space [3] are obtained practically for $\lambda = 5$.

The dependence of the ratio P_{H}/P on the parameter θ at fixed values of the layer thickness γ is shown in Fig. 3 by the continuous lines. This ratio decreases with the parameter θ rising, i.e. with the growth of the thermal effects. In this same manner such for the half-space ($\lambda = \infty$) these dependencies may be approximated with the very good accuracy (maximum deviation $< 1\%$) in the forms of linear relation

$$\frac{P_{\mathrm{H}}}{P} + \left(\frac{\theta}{\theta^*} - 1\right)\frac{P_{\mathrm{H}}}{P_0} = 0 \quad (34)$$

where θ^* is the limit value of the parameter θ as the load P tends to infinity and P_0 is the value of the load in the isothermal problem for the layer. The parameters θ^* and P_0 depends on the thickness λ. The results for $\lambda = 5$ are closed to the solution for the half-space [2] $\theta^* = \theta_\infty = 1.18$. The decreasing of the layer thickness causes the limit value of the parameter θ to decrease, i.e. the load P, which is needed to obtain the contact area $a_{\mathrm{H}} = a$, rises. The formula (34) gives

$$\frac{P_0}{P} + \frac{\theta}{\theta^*} - 1 = 0. \quad (35)$$

This dependence at the fixed values of the layer thickness is shown in Fig. 3 by the dotted lines.

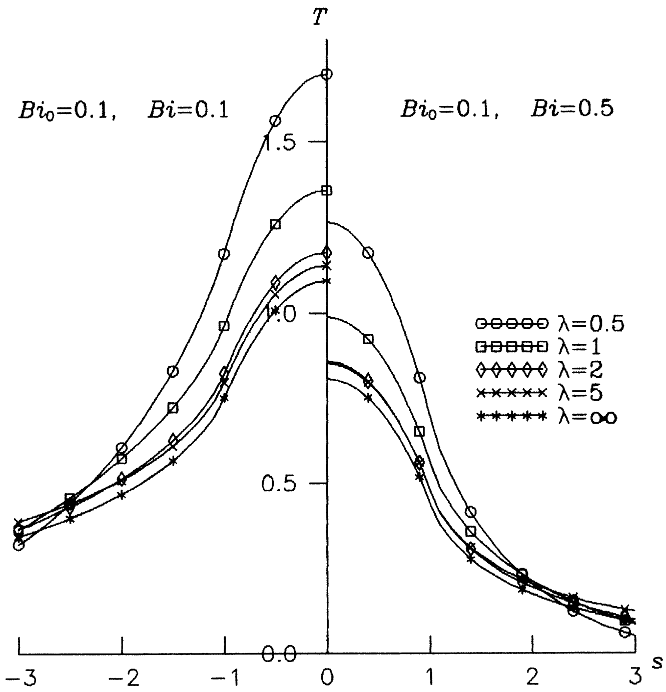

Fig. 2. Effect of the thickness λ on the dimensionless surface temperature.

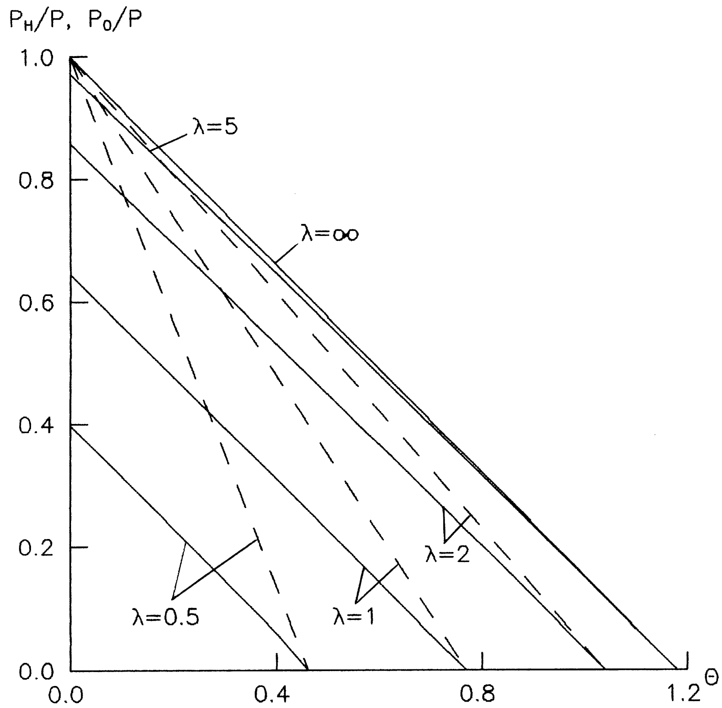

Fig. 3. Ratios P_H/P and P_0/P vs. the parameter θ.

The numerical analysis shows that the Biot's coefficients Bi and Bi_0 do not affect the ratio P_H/P and therefore the contact semi-width.

Figure 4 shows the dependence of the ratio P_H/P on the layer thickness λ for some values of the parameter θ. The decreasing of the layer thickness as well as the rising of the parameter θ leads to the falling of the ratio P_H/P.

The effect of the layer thickness on the limit value of the parameter θ is shown in Fig. 5 by the dotted line. These dependencies may be closely approximated by the formula

$$\theta^* = \theta_\infty - \frac{\theta_1}{\lambda_1} - \frac{\theta_2}{\lambda^2} - \frac{\theta_3}{\lambda^3}, \quad \lambda > 0.25 \tag{36}$$

where $\theta_1 = 0.069$, $\theta_2 = 0.215$, $\theta_3 = -0.044$. In Fig. 5 the continuous line correspond to the function (36). The mean relative accuracy of this approximation is equal to 2.7%. Note that the terms with $1/\lambda^2$ and $1/\lambda^3$ can be rejected in (36) in the problem for a thick layer.

Using the formula (26) for the θ the limit value of the contact semi-width may be presented in the form

$$a_{cr} = \frac{(1-v)\theta^*}{fV\mu\delta}$$

or, taking into account the formula (36)

$$a_{cr}^* = a_{cr}^\infty \left[1 - \frac{a_1}{\lambda} - \frac{a_2}{\lambda^2} - \frac{a_3}{\lambda^3}\right], \quad \lambda > 0.25 \tag{37}$$

where $a_{cr}^\infty = [(1-v)\theta_\infty/fV\mu\delta]$ is the limit value of the contact semi-width for the half-space, and

$a_1 = 0.058, \quad a_2 = 0.182, \quad a_3 = -0.037.$

Thus, the contact area in the problem for the layer is less that in the problem for the half-space.

7. Conclusions

The contact problem for the layer involving frictional heating was solved. The effect of the heat generation on the contact temperature and on the contact semi-width was studied. The comparison of the results obtained with that for the half-space permits us to make the following conclusions:

—the temperature generated in the contact of the rigid cylinder and the elastic layer is higher than that in the contact of this cylinder and the half-space;
—the thermal effects produced by the frictional heating are more important here than in the case of the half-space. The contact region in this problem is smaller than that in the corresponding problem for the half-space;
—the approximate formula to compare the contact region size in the cases of the layer and the half-space was obtained. The limit values of the contact semi-

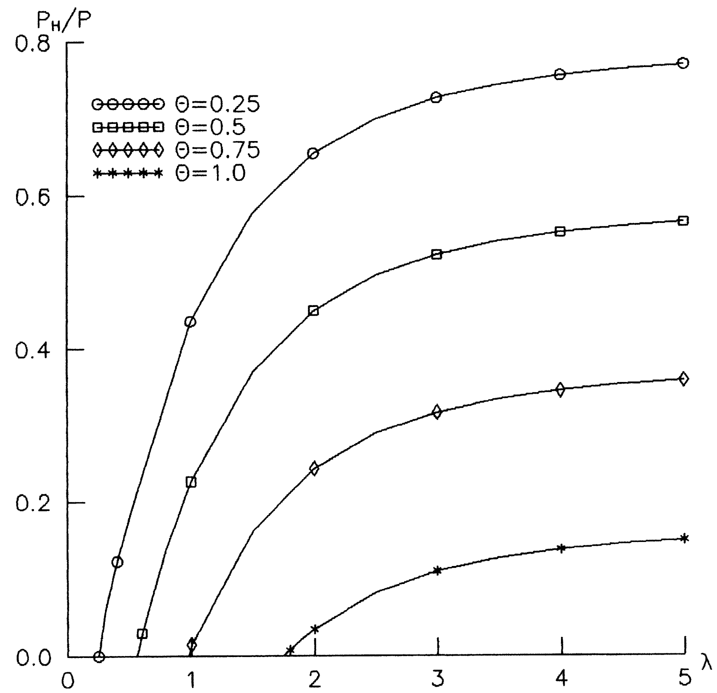

Fig. 4. Ratio P_H/P vs. the thickness λ.

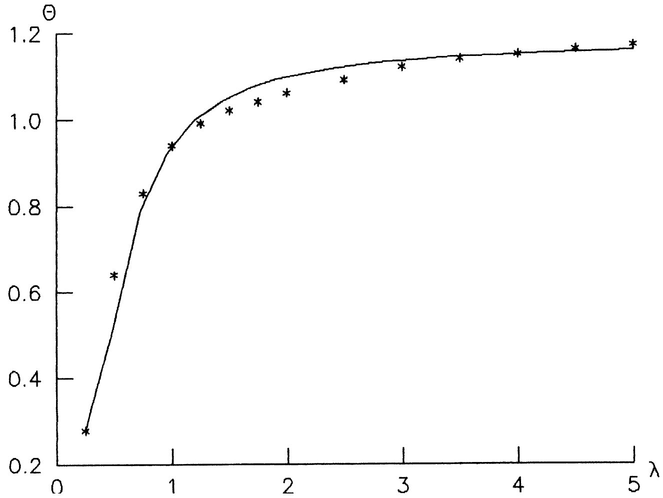

Fig. 5. Effect of the thickness λ on the limit value of the parameter θ.

width are reached for the smaller values of the input thermomechanical parameter θ.

These results can be used to study thermal regimes of the real contact pairs.

Appendix

Let the function $g(\zeta)$ decay exponentially for $\zeta \to \infty$. We approximate this function by a function $\hat{g}(\zeta)$, which is defined as

$$\hat{g}(\zeta) = \sum_{m=1}^{M} a_m \, e^{-m\beta\zeta} \quad \text{(A1)}$$

where the constants a_m, $m = 1, \ldots, M$ and β are determinated by the summation of the squared error

$$S = \sum_{l=1}^{L} [g(\zeta_l) - \hat{g}(\zeta_l)]^2. \quad \text{(A2)}$$

The minimizing condition

$$\frac{\partial S}{\partial a_k} = 0, \quad k = 1, \ldots, M \quad \text{(A3)}$$

gives the system of equations for the determination of the constants a_m, $m = 1, \ldots, M$

$$\sum_{m=1}^{M} a_m \left\{ \sum_{l=1}^{L} \exp\left[-(m+k)\beta\zeta_l\right] \right\} = \sum_{l=1}^{L} g(\zeta_l) \exp(-k\beta\zeta_l), \quad k = 1, \ldots, M \quad \text{(A4)}$$

which can be simplified if the points ζ_l, $l = 1, \ldots, L$ are chosen as

$$\zeta_l = l\zeta_{\max}/L, \quad l = 1, \ldots, L. \quad \text{(A5)}$$

Thus

$$\sum_{l=1}^{L} \exp\left[-(m+k)\beta\zeta_l\right] = \sum_{l=1}^{L} a_{mk}^{l} = \frac{1 - a_{mk}^{L+1}}{1 - a_{mk}}, \quad k = 1, \ldots, M \quad \text{(A6)}$$

where

$$a_{mk} = \exp\left[-(m+k)\beta\zeta_{\max}/L\right] \quad \text{(A7)}$$

and the system (A4) can be transformed

$$\sum_{m=1}^{M} a_m \left\{ \frac{1 - a_{mk}^{L+1}}{1 - a_{mk}} \right\} = \sum_{l=1}^{L} g(\zeta_l) \exp(-k\beta\zeta_l), \quad k = 1, \ldots, M. \quad \text{(A8)}$$

The constant β is determined iteratively from the condition of the value S minimum. The accuracy of this method is provided by the choice of constants M, L and $\zeta_{\max}$.

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